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On algebraic classification of quasi-exactly solvable matrix models

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Abstract. We suggest a generalization of the Lie algebraic approach for constructing quasi-exactly solvable one-dimensional Schrödinger equations. This generalization is based on representations of Lie algebras by first-order matrix differential operators. We have classified inequivalent representations of the Lie algebras of dimensions up to three by first-order matrix differential operators in one variable. Next we describe invariant finite-dimensional subspaces of the representation spaces of the one-, two-dimensional Lie algebras and of the algebra $sl(2, \mathbb{R})$. These results enable us to construct multiparameter families of first- and second-order quasi-exactly solvable models. In particular, we have obtained two classes of quasi-exactly solvable matrix Schrödinger equations.

1. Introduction

There exists a small number of remarkable Hamiltonians (called exactly solvable) whose spectra and corresponding eigenfunctions can be computed in a purely algebraic way (see, e.g. [1]). However, the choice of such Hamiltonians is too restricted to meet numerous requirements coming from different fields of modern quantum physics. Recently, an intermediate class of Hamiltonians was introduced by Turbiner [2] and Ushveridze [3] which allows an algebraic construction of the part of their spectra. Spectral problems of this kind are called *quasi-exactly solvable*.

Quasi-exactly solvable models have an amazingly wide range of applications in different fields of theoretical physics including conformal quantum-field theories [4], solid-state physics [5, 6] and Gaudin algebras (an excellent survey on this subject and an extensive list of references can be found in [7]). So it was only natural that there appeared different approaches to constructing quasi-exactly solvable models, including the one based on their conditional symmetries (for more details see [8]). However, for the purposes of this paper the most appropriate is the Lie-algebraic approach suggested by Shifman [9], Shifman and Turbiner [10] and further developed by González-López *et al* [11–13]. That is why we will give its brief description (further details can be found in [7]).

The Lie-algebraic approach to constructing quasi-exactly solvable one-dimensional stationary Schrödinger equations

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right)\psi(x) = \lambda\psi(x) \tag{1}$$

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heavily relies upon the properties of representations of the algebra $sl(2, \mathbb{R})$

$$[Q_0, Q_\pm] = \pm Q_\pm \qquad [Q_-, Q_+] = 2Q_0$$

by first-order differential operators. Namely, the approach in question utilizes the fact that the representation space of the algebra $sl(2, \mathbb{R})$ having the basis elements

$$Q_- = \frac{\mathrm{d}}{\mathrm{d}x} \qquad Q_0 = x\frac{\mathrm{d}}{\mathrm{d}x} - \frac{n}{2} \qquad Q_+ = x^2\frac{\mathrm{d}}{\mathrm{d}x} - nx \tag{2}$$

where n is an arbitrary natural number, has an $(n+1)$-dimensional invariant subspace. Its basis is formed by the polynomials in x of an order not higher than n. Due to this fact, any bilinear combination of operators (2) with constant coefficients yields a quasi-exactly solvable Hamiltonian $\mathcal{H}$ such that the equation $\mathcal{H}\psi = \lambda\psi$ can always be reduced to the form (1) with the help of a transformation

$$\psi(x) \to F(x)\tilde{\psi}(f(x)).$$

Note that the above-described procedure does not guarantee that eigenfunctions of thus constructed quasi-exactly solvable Hamiltonians will be square-integrable. What can be done within this approach is to reduce a 'differential' eigenvalue problem to a matrix one. The matter of a square integrability as well as other analytical properties of solutions obtained are to be investigated separately by independent methods (see, e.g. [14]).

Recently, a number of papers devoted to constructing matrix quasi-exactly solvable models have been published [15–18]. These papers use the same basic idea which is to fix a concrete subspace of sufficiently smooth multicomponent functions and then to classify all second-order matrix differential operators leaving this subspace invariant. Furthermore, the above subspace is chosen to be the space of all multicomponent functions with polynomial components. Posed in this way, the problem of constructing matrix quasi-exactly solvable includes as a subproblem, one of classifying realizations of Lie superalgebras by differential operators. Being very rich in interesting and important results, this approach, however, contains an evident restriction which does not allow us to construct *all* possible quasi-exactly solvable models. What we mean is, the fact that an invariant subspace is not necessarily spanned by functions having polynomial coefficients. It is one of the results of this paper that there exist principally different invariant subspaces. Thus, there is a necessity for developing alternative approaches to the problem in question that do not require fixing *a priori* an invariant subspace.

Our initial motivation for studying matrix quasi-exactly solvable problems was to extend the list of exactly solvable Dirac equations of an electron via a separation of variables. To this end in [19] we suggested a method for constructing matrix quasi-exactly solvable models based on a direct generalization of the Lie-algebraic approach for the case of multicomponent wavefunctions. However, it turns out that the above method is universal enough to be applied for obtaining second-order quasi-exactly solvable models as well, including the Schrödinger equations with matrix potentials.

Following [19] we extend the class to which should belong basis elements of a Lie algebra under study (say, of the algebra $sl(2, \mathbb{R})$). We define this class as the set of matrix differential operators

$$Q = \xi(x)\frac{\mathrm{d}}{\mathrm{d}x} + \eta(x) \tag{3}$$

where $\xi(x)$ is a smooth real-valued function, $\eta(x)$ is a smooth complex-valued $r \times r$ matrix function, and denote it as $\mathcal{M}$. The class $\mathcal{M}$ is closed with respect to the binary operation

$$\{Q_1, Q_2\} \to Q_1Q_2 - Q_2Q_1 \stackrel{\text{def}}{=} [Q_1, Q_2]$$

and, consequently, forms the infinite-dimensional Lie algebra.

We will classify inequivalent representations of low-dimensional ($d \leqslant 3$) Lie algebras by operators belonging to $\mathcal{M}$. Next, we will study the additional constraints on the form of basis operators of the one- and two-dimensional Lie algebras imposed by the requirement that their representation spaces contain finite-dimensional invariant subspaces. These results will be used to obtain an exhaustive description of inequivalent representations of the algebra $sl(2,\mathbb{R})$ by matrix differential operators (3) with $r = 2$. Composing linear and bilinear combinations of basis elements of $sl(2,\mathbb{R})$ with constant matrix coefficients will yield multiparameter first- and second-order quasi-exactly solvable matrix models.

2. Classification of representations of low-dimensional Lie algebras

Since our aim is to get a quasi-exactly solvable model, we have to impose an additional restriction on the choice of the basis elements of the Lie algebras to be considered below. Namely, it is supposed that there exists at least one basis element such that the coefficient of $\mathrm{d}/\mathrm{d}x$ does not vanish identically. This constraint is required to avoid purely matrix representations which are useless in context of quasi-exactly solvable models.

Consider a first-order differential operator $Q = \xi(x)\partial_x + \eta(x)$ with $\xi \not\equiv 0$. Note that hereafter we denote $\mathrm{d}/\mathrm{d}x$ as ∂_x. Let the function $f(x)$ be defined by the relation

$$f(x) = \int_a^x \frac{\mathrm{d}y}{\xi(y)} \qquad a \in \mathbb{R}$$

and the matrix $r \times r$ function $F(x)$ be a solution of system of ordinary differential equations

$$\xi(x)\frac{\mathrm{d}F(x)}{\mathrm{d}x} + \eta(x)F(x) = 0$$

with $\det F(x) \neq 0$. Then the equivalence transformation

$$Q \to \tilde{Q} = (F(x))^{-1}QF(x)$$

with a subsequent change of the dependent variable

$$\tilde{x} = f(x)$$

reduces the operator Q to become $\tilde{Q} = \partial_{\tilde{x}}$. Consequently, any one-dimensional Lie algebra of first-order matrix differential operators $Q = \xi(x)\partial_x + \eta(x)$ with $\xi \not\equiv 0$ is equivalent to the algebra $\langle\partial_x\rangle$.

Abstract Lie algebras of dimensions up to five have been classified by Mubarakzyanov [20]. Below we give the lists of (non-zero) commutation relations which determine inequivalent Lie algebras of dimensions up to three. Note that the algebras which are direct sums of lower-dimensional Lie algebras are omitted from the lists.

$$\begin{array}{llll}
L_{2,1}: & [Q_1, Q_2] = Q_1 & & \\
L_{3,1}: & [Q_2, Q_3] = Q_1 & & \\
L_{3,2}: & [Q_1, Q_3] = Q_1 & [Q_2, Q_3] = Q_1 + Q_2 & \\
L_{3,3}: & [Q_1, Q_3] = Q_1 & [Q_2, Q_3] = Q_2 & \\
L_{3,4}: & [Q_1, Q_3] = Q_1 & [Q_2, Q_3] = -Q_2 & \\
L_{3,5}: & [Q_1, Q_3] = Q_1 & [Q_2, Q_3] = aQ_2 & (0 < |a| < 1) \\
L_{3,6}: & [Q_1, Q_3] = -Q_2 & [Q_2, Q_3] = Q_1 & \\
L_{3,7}: & [Q_1, Q_3] = aQ_1 - Q_2 & [Q_2, Q_3] = Q_1 + aQ_2 & (a > 0)
\end{array}$$

$$L_{3,8}: \quad [Q_1, Q_2] = Q_1 \qquad [Q_1, Q_3] = 2Q_2 \qquad [Q_2, Q_3] = Q_3$$
$$L_{3,9}: \quad [Q_1, Q_2] = Q_3 \qquad [Q_2, Q_3] = Q_1 \qquad [Q_3, Q_1] = Q_2.$$

Here a is a real parameter, the symbol $L_{n,m}$ stands for a Lie algebra of dimension n numbered by m.

Thus, there exists only one two-dimensional Lie algebra $L_{2,1} = \langle Q_1, Q_2\rangle$ which is not a direct sum of one-dimensional Lie algebras.

If in the operator $Q_1 = \xi(x)\partial_x + \eta(x)$ the coefficient ξ is not identically zero, then using equivalence transformations defined at the beginning of this section we can reduce it to the form $Q_1 = \partial_{\tilde{x}}$. Inserting $Q_1 = \partial_{\tilde{x}}$, $Q_2 = \tilde{\xi}(\tilde{x})\partial_{\tilde{x}} + \tilde{\eta}(\tilde{x})$ into the commutation relation $[Q_1, Q_2] = Q_1$ and equating the coefficients of the powers of the operator $\partial_{\tilde{x}}$ yield systems of ordinary differential equations for $\tilde{\xi}(\tilde{x})$, $\tilde{\eta}(\tilde{x})$

$$\frac{\mathrm{d}\tilde{\xi}}{\mathrm{d}\tilde{x}} = 1 \qquad \frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}\tilde{x}} = 0.$$

Hence we obtain $\tilde{\xi} = \tilde{x} + C_1$, $\tilde{\eta} = A$, where $C_1 \in \mathbb{R}$ is an arbitrary constant and A is an arbitrary constant $r \times r$ matrix. Shifting the variable $\tilde{x}$ when necessary by a constant C_1 we may put $C_1 = 0$ and thus get $Q_2 = \tilde{x}\partial_{\tilde{x}} + A$.

If the operator Q_1 has the form $\eta(x)$, then by convention the coefficient of ∂_x of the operator Q_2 does not vanish identically. Consequently, there exists an equivalence transformation reducing the latter to the form $Q_2 = \partial_{\tilde{x}}$. Substituting $Q_1 = \tilde{\eta}(\tilde{x})$, $Q_2 = \partial_{\tilde{x}}$ into the commutation relation of the algebra $L_{2,1}$ and equating the coefficients of the powers of the operator ∂_x we obtain the following equation for $\tilde{\eta}(x)$:

$$\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}\tilde{x}} = -\tilde{\eta}$$

whence

$$\tilde{\eta}(\tilde{x}) = A\mathrm{e}^{-\tilde{x}}.$$

Here A is an arbitrary $r \times r$ constant matrix.

Therefore, we conclude that the two realizations of the algebra $L_{2,1}$

$$(1) \qquad Q_1 = A\mathrm{e}^{-x} \qquad Q_2 = \partial_x \tag{4}$$
$$(2) \qquad Q_1 = \partial_x \qquad Q_2 = x\partial_x + A \tag{5}$$

exhaust the set of all possible inequivalent representations of the algebra in question within the class of matrix differential operators $\mathcal{M}$.

In a similar way we have obtained complete lists of inequivalent representations of the three-dimensional Lie algebras within the class $\mathcal{M}$ which are given below.

$$L_{3,1}: Q_1 = A \qquad Q_2 = \partial_x \qquad Q_3 = \epsilon\partial_x + Ax + B$$
$$[A, B] = 0$$
$$L_{3,2}: Q_1 = A\mathrm{e}^{-x} \qquad Q_2 = \epsilon\mathrm{e}^{-x}\partial_x + (B - Ax)\mathrm{e}^{-x} \qquad Q_3 = \partial_x$$
$$[A, B] = -\epsilon A$$
$$L_{3,3}: Q_1 = \mathrm{e}^{-x}(\epsilon\partial_x + A) \qquad Q_2 = \mathrm{e}^{-x}(\alpha\partial_x + B) \qquad Q_3 = \partial_x$$
$$[A, B] = \epsilon B - \alpha A$$
$$L_{3,4}:$$
$$(1) \quad Q_1 = A\mathrm{e}^{-x} \qquad Q_2 = \mathrm{e}^{x}(\epsilon\partial_x + B) \qquad Q_3 = \partial_x$$
$$[A, B] = -\epsilon A$$

(2) $Q_1 = \partial_x \qquad Q_2 = A \qquad Q_3 = x\partial_x + B$

$[A, B] = -A$

$L_{3,5}$:

(1) $Q_1 = A\mathrm{e}^{-x} \qquad Q_2 = \mathrm{e}^{-ax}(\epsilon\partial_x + B) \qquad Q_3 = \partial_x$

$[A, B] = -\epsilon A$

(2) $Q_1 = \mathrm{e}^{-x}(\partial_x + A) \qquad Q_2 = B\mathrm{e}^{-ax} \qquad Q_3 = \partial_x$

$[A, B] = \epsilon a B$

$L_{3,6}$: $Q_1 = A\cos x + B\sin x \qquad Q_2 = B\cos x - A\sin x \qquad Q_3 = \partial_x$

$[A, B] = 0$

$L_{3,7}$: $Q_1 = \mathrm{e}^{-ax}(A\cos x + B\sin x) \qquad Q_2 = \mathrm{e}^{-ax}(B\cos x - A\sin x)$

$Q_3 = \partial_x$

$[A, B] = 0$

$L_{3,8}$: $Q_1 = \partial_x \qquad Q_2 = x\partial_x + A \qquad Q_3 = x^2\partial_x + 2Ax + B$

$[A, B] = B$

$L_{3,9}$: No representations.

In the above formulae α is an arbitrary constant, $\epsilon = 0, 1$, and A, B are $r \times r$ constant matrices.

3. Quasi-exactly solvable matrix models

As a second step of an implementation of the Lie-algebraic approach to constructing matrix quasi-exactly solvable models we have to pick out from the whole set of realizations of Lie algebras listed in the previous section those having finite-dimensional invariant subspaces.

Consider first the one-dimensional Lie algebra $\langle\partial_x\rangle$. A space with basis vectors $\boldsymbol{f}_1(x), \ldots, \boldsymbol{f}_n(x)$ is invariant with respect to the action of the operator ∂_x if there exist complex constants Λ_{ij} such that

$$\frac{\mathrm{d}\boldsymbol{f}_i(x)}{\mathrm{d}x} = \sum_{j=1}^{n} \Lambda_{ij}\boldsymbol{f}_j(x)$$

for all $i = 1, \ldots, n$. Solving this system of ordinary differential equations yields the following expressions for unknown vector functions $\boldsymbol{f}_i$:

$$\boldsymbol{f}_i(x) = \sum_{j=1}^{r}\sum_{k=1}^{n} (\mathrm{e}^{\Lambda x})_{ik} C_{kj}\boldsymbol{e}_j$$

where Λ is the constant $n \times n$ matrix having the entries Λ_{ij}; C_{kj} are arbitrary complex constants; the symbol $(A)_{ij}$ stands for the (i, j)th entry of the matrix A and $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_r$ are constant vectors forming an orthonormal basis of the space $\mathbb{R}^r$.

It follows from the general theory of matrices that the above formulae can be represented in the following equivalent form (see, e.g. [21]):

$$\boldsymbol{f}_i(x) = \sum_{j=1}^{m} \mathrm{e}^{\alpha_j x} \sum_{k=1}^{n} \mathcal{P}_{ijk}^{[n-m]}(x)\boldsymbol{e}_k. \tag{6}$$

Here $\alpha_1, \ldots, \alpha_m$ are arbitrary complex numbers with $|\alpha_i| < |\alpha_{i+1}|$, the symbol $\mathcal{P}_{ijk}^{[n-m]}(x)$ stands for an $(n-m)$th degree polynomial in x, $1 \leqslant m \leqslant n$, $k = 1, \ldots, n$.

As each realization of the low-dimensional Lie algebras obtained in section 2 contains the operator ∂_x, their finite-dimensional invariant subspaces are necessarily of the form (6). In what follows we will obtain a complete description of finite-dimensional invariant subspaces of the representation spaces of the representations of the two-dimensional Lie algebra $L_{2,1}$ given in (4) and (5).

First we turn to the case (4). Let us study the restrictions on the choice of the basis vector functions (6) imposed by a requirement that the corresponding vector space V_n is invariant with respect to the action of the operator $Q_2 = A\mathrm{e}^{-x}$. By assumption, there exist complex constants S_{ij} such that the relations

$$A\mathrm{e}^{-x}\boldsymbol{f}_i = \sum_{j=1}^{n} S_{ij}\boldsymbol{f}_j$$

hold with $i = 1, \ldots, n$. Hence we get

$$\sum_{j=1}^{m} \mathrm{e}^{(\alpha_j-1)x} \sum_{k=1}^{r} \mathcal{P}_{ijk}^{[n-m]}(x)A\boldsymbol{e}_k = \sum_{j=1}^{m} \mathrm{e}^{\alpha_j x} \sum_{k=1}^{r} \tilde{\mathcal{P}}_{ijk}^{[n-m]}(x)\boldsymbol{e}_k$$

where

$$\tilde{\mathcal{P}}_{ijk}^{[n-m]} = \sum_{l=1}^{n} S_{il}\mathcal{P}_{ljk}^{[n-m]}.$$

Comparing the coefficients of $\mathrm{e}^{\alpha_i x}$ yield that $\alpha_{i+1} = \alpha_i + 1$, $i = 1, \ldots, m-1$ and furthermore

$$\sum_{k=1}^{r} \mathcal{P}_{i1k}^{[n-m]}(x)A\boldsymbol{e}_k = 0 \tag{7}$$

for all $i = 1, \ldots, n$.

Let us choose the new basis of the space $\mathbb{R}^r$ in such a way that the first s basis elements $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_s$ are eigenvectors of the matrix A with zero eigenvalues, namely

$$A\boldsymbol{e}_i = \mathbf{0} \qquad i = 1, \ldots, s.$$

Given this choice of the basis, it follows from (7) that $\mathcal{P}_{i1k}^{[n-m]}(x) = 0, k = s+1, \ldots, r$. Hence, we conclude that the remaining basis vectors $\boldsymbol{e}_{s+1}, \ldots, \boldsymbol{e}_r$ satisfy the relations

$$A\boldsymbol{e}_i = \sum_{j=1}^{s} a_{ij}\boldsymbol{e}_j \qquad i = s+1, \ldots, r$$

with some constant a_{ij}.

Thus, the most general n-dimensional vector space V_n invariant with respect to the Lie algebra $\langle \partial_x, A\mathrm{e}^{-x}\rangle$ is spanned by the vectors

$$\boldsymbol{f}_i(x) = \mathrm{e}^{\alpha x} \sum_{j=2}^{m} \mathrm{e}^{(j-1)x} \sum_{k=1}^{r} \mathcal{P}_{ijk}^{[n-m]}(x)\boldsymbol{e}_k + \mathrm{e}^{\alpha x} \sum_{j=1}^{s} \mathcal{P}_{ij}^{[n-m]}(x)\boldsymbol{e}_j$$

where α is an arbitrary complex constant, $\mathcal{P}_{ijk}^{[n-N]}(x), \mathcal{P}_{ij}^{[n-N]}(x)$ are arbitrary $(n-N)$th-order polynomials in x, $i = 1, \ldots, n$. And what is more the matrix A is of the following form:

$$A = \begin{pmatrix} 0 & \tilde{A} \\ 0 & 0 \end{pmatrix}$$

where $\tilde{A}$ is an arbitrary constant $s \times (r-s)$ matrix.

Now we turn to representation (5). It is necessary to investigate the restrictions on the choice of the basis vector functions (6) imposed by a requirement that the corresponding vector space V_n is invariant with respect to the action of the operator $Q_2 = x\partial_x + A$. By assumption, there exist complex constants S_{ij} such that the relations

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x} + A\right)\boldsymbol{f}_i = \sum_{j=1}^{n} S_{ij}\boldsymbol{f}_j$$

hold with $i = 1, \ldots, n$. Inserting expressions (6) into these equations and comparing the coefficients of $\mathrm{e}^{\alpha_j x}x^k$ we arrive at the conclusion that $\alpha_1 = \cdots = \alpha_m = 0$. With this restriction formulae (6) give the most general finite-dimensional invariant subspace of the representation space of the Lie algebra $\langle \partial_x, x\partial_x + A\rangle$

$$\boldsymbol{f}_i(x) = \sum_{k=1}^{r} \mathcal{P}_{ik}^{[n-1]}(x)\boldsymbol{e}_k \qquad i = 1, \ldots, n.$$

A detailed investigation of finite-dimensional invariant subspaces admitted by the three- and four-dimensional Lie algebras is in progress now and will be the topic of our future publications.

In what follows we will construct examples of quasi-exactly solvable two-component matrix models based on representations of the Lie algebra $L_{3,8} = sl(2, \mathbb{R})$. The construction procedure relies upon the assertion below which is given without proof.

Theorem 1. The representation space of the algebra $sl(2, \mathbb{R})$ having the basis elements

$$Q_1 = \partial_x \qquad Q_2 = x\partial_x + A \qquad Q_2 = x^2\partial_x + 2xA + B \tag{8}$$

where A and B are constant 2×2 matrices satisfying the relation $[A, B] = B$, contains a finite-dimensional invariant subspace iff the matrices A, B are of the form

$$(1) \qquad A = \begin{pmatrix} -\frac{n}{2} & 0 \\ 0 & -\frac{m}{2} \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{9}$$

or

$$(2) \qquad A = \begin{pmatrix} -\frac{n}{2} & 0 \\ 0 & \frac{2-n}{2} \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{10}$$

Here n and m are arbitrary natural numbers with $n \geqslant m$.

Using the fact that the algebra in question has the Casimir operator $C = B\partial_x + A - A^2$ it is not difficult to become convinced of the fact that representations of the form (8), (10) are the direct sums of two irreducible representations realized on the representation spaces

$$\mathcal{R}_1 = \langle \boldsymbol{e}_1, x\boldsymbol{e}_1, \ldots, x^n\boldsymbol{e}_1\rangle \qquad \mathcal{R}_2 = \langle \boldsymbol{e}_2, x\boldsymbol{e}_2, \ldots, x^m\boldsymbol{e}_2\rangle$$

where $\boldsymbol{e}_1 = (1, 0)^{\mathrm{T}}$, $\boldsymbol{e}_2 = (0, 1)^{\mathrm{T}}$.

Representations (8), (10) are also the direct sums of two irreducible representations realized on the representation spaces

$$\mathcal{R}_1 = \langle n\boldsymbol{e}_1, \ldots, nx^j\boldsymbol{e}_1 + jx^{j-1}\boldsymbol{e}_2, \ldots, nx^n\boldsymbol{e}_1 + nx^{n-1}\boldsymbol{e}_2\rangle$$
$$\mathcal{R}_2 = \langle \boldsymbol{e}_2, x\boldsymbol{e}_2, \ldots, x^{n-2}\boldsymbol{e}_2\rangle.$$

According to the scheme given in the introduction to obtain a quasi-exactly solvable model we have to compose a linear combination of basis elements of a Lie algebra of differential operators whose representation space has finite-dimensional invariant subspace V_n. And what is more, coefficients of this linear combination are constant matrices of the corresponding dimension whose action leaves the space V_n invariant.

Consider first the representation (8), (10) with $n = m$. According to the above its representation space contains $2(n+1)$-dimensional invariant subspace V_{2n+2} spanned by the vectors $\boldsymbol{e}_1 x^j, \boldsymbol{e}_2 x^j,\ j = 0, \ldots, n$. A direct verification shows that V_{2n+2} is invariant with respect to action of any 2×2 matrix. Composing a linear combination of (8), (10) under $n = m$ with coefficients being arbitrary 2×2 matrices yields the following quasi-exactly solvable model:

$$(A_1 + A_2 x + A_3 x^2)\frac{\mathrm{d}\boldsymbol{u}(x)}{\mathrm{d}x} + (A_4 - nxA_3)\boldsymbol{u} = \lambda\boldsymbol{u}. \tag{11}$$

Here A_1, A_2, A_3, A_4 are arbitrary constant 2×2 matrices, $\boldsymbol{u}(x)$ is a two-component vector function.

Provided $n = m+1$, representation (8), (10) gives rise to a quasi-exactly solvable matrix model of the form

$$(A_1 + B_1 x + B_2 x^2)\frac{\mathrm{d}\boldsymbol{u}(x)}{\mathrm{d}x} + (B_3 + B_2 A)\boldsymbol{u} = \lambda\boldsymbol{u} \tag{12}$$

where A_1 is an arbitrary 2×2 matrix and B_1, B_2, B_2 are arbitrary upper triangular 2×2 matrices.

Finally, if $n > m+1$, then representation (8), (10) yields a quasi-exactly solvable matrix model of the form (12), where both A_1 and B_1, B_2, B_3 are arbitrary upper triangular 2×2 matrices.

A similar analysis shows that representation (8), (10) gives rise to a quasi-exactly solvable model

$$(B_1 + B_2 x + B_3 x^2)\frac{\mathrm{d}\boldsymbol{u}(x)}{\mathrm{d}x} + (B_4 + B_3(2xA + B))\boldsymbol{u} = \lambda\boldsymbol{u} \tag{13}$$

where

$$B_i = \begin{pmatrix} \lambda_i & b_i \\ 0 & \lambda_i \end{pmatrix} \qquad i = 1, \ldots, 4$$

λ_i, b_i being arbitrary complex constants.

Needless to say that any linear matrix model obtained from one of the above quasi-exactly solvable models by a change of variables is in its turn quasi-exactly solvable. In other words, equations (11)–(13) are representatives of equivalence classes of quasi-exactly solvable models. Other representatives are obtained via a transformation of variables

$$x \to \tilde{x} = f(x) \qquad \boldsymbol{u} \to F(x)\tilde{\boldsymbol{u}} \tag{14}$$

where $f(x)$ is a smooth function and $F(x)$ is an arbitrary invertible 2×2 matrix whose entries are smooth functions of x.

In the same way, second-order quasi-exactly solvable matrix models are constructed. In particular, taking a bilinear combination of operators (8) with A, B being given either by (9) or (10)

$$\mathcal{H} = \sum_{i,j=1, i \leqslant j}^{3} \alpha_{ij} Q_i Q_j + \sum_{i=1}^{3} \alpha_j Q_j$$

where α_{ij}, α_i are arbitrary real constants, yields two families of quasi-exactly solvable matrix models of the form

$$\mathcal{H}\boldsymbol{u} = \lambda\boldsymbol{u}.$$

By a suitable transformation (14) the latter can be transformed to become a matrix Schrödinger equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\tilde{x}^2} + V(\tilde{x})\right)\tilde{\boldsymbol{u}} = \lambda\tilde{\boldsymbol{u}}.$$

Here $V(\tilde{x})$ is the 2×2 matrix potential, whose explicit form depends essentially on the parameters α_{ij}, α_i and integers n, m. Let us stress that the matter of hermiticity of thus obtained matrix potential is by no means clear and needs special investigation.

With a particular choice of the matrices A, B

$$A = \begin{pmatrix} -\frac{n}{2} & 0 \\ 0 & -\frac{n}{2} \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

the well known nine-parameter family of scalar quasi-exactly solvable Schrödinger equations is obtained [2–7].

4. Conclusions

This paper is primarily aimed at solving the problem of the classification of quasi-exactly matrix models by purely algebraic means. As a result, we obtain some classes of systems of first- and second-order ordinary differential equations such that a problem of finding their particular solutions reduces to solving the *matrix* eigenvalue problem. Now to decide whether a given specific matrix model is solvable within the framework of an approach expounded above, one has to check whether it is possible to reduce it with the help of transformation (14) to one of the canonical forms given in section 3. When one deals with a scalar model this check is done trivially (see [7] for details). However, for the case of matrix models it involves tedious and cumbersome calculations and is by itself rather a non-trivial algebraic problem. As an illustration we will adduce an instructive example. Consider the following two-component matrix model:

$$\mathcal{H}\boldsymbol{u} \equiv (\mathrm{i}b\sigma_1 Q_1 + \mathrm{i}a\sigma_2 Q_2 + c_1\sigma_1 + c_2\sigma_2)\boldsymbol{u} = \lambda \boldsymbol{u} \qquad (15)$$

where a, b, c_1, c_2 are arbitrary real parameters with $ab \neq 0$, and σ_1, σ_2 are 2×2 Pauli matrices, and

$$Q_1 = \partial_x \qquad Q_2 = x\partial_x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This model is quasi-exactly solvable by construction. Making the change of variables

$$x = \frac{b}{a}\sinh(ay)$$

$$\boldsymbol{w}(x) = (\cosh(ay))^{1/2} \exp\left\{-\frac{\mathrm{i}}{a}(c_1 \arctan\sinh(ay) + c_2 \ln\cosh(ay))\right\}$$
$$\times \exp\{-\mathrm{i}\sigma_3 \arctan\sinh(ay)\}\boldsymbol{\psi}(y)$$

we reduce (15) to the Dirac-type equation

$$\mathrm{i}\sigma_1 \frac{\mathrm{d}\boldsymbol{\psi}}{\mathrm{d}y} + \sigma_2 V(y)\boldsymbol{\psi}(y) = \lambda\boldsymbol{\psi} \qquad (16)$$

where

$$V(y) = \frac{a^2c_2 - b^2c_1\sinh(ay)}{ab\cosh(ay)}$$

is the well known hyperbolic Pöschl–Teller potential. This means that the Dirac equation (16) with the hyperbolic Pöschl–Teller potential is quasi-exactly solvable.

Thus, applying the obtained results to decide whether a given model is quasi-exactly solvable requires considerable experience in manipulating matrix exponents. Generically, to check which equations of the form (16) can be reduced to one of the quasi-exactly solvable models constructed at the end of section 3 one has to solve systems of nonlinear algebraic equations.

A technique used in this paper can be generalized in order to enable one to classify multidimensional matrix models in the same way as done for a scalar case by González-López *et al* [11–13].

The final important remark is that the property of quasi-exact solvability is intimately connected to the conditional symmetry of a model under study. This fact was first noticed in [8], where we proved that the quasi-exact solvability of stationary Schrödinger equations is in one-to-one correspondence with their conditional symmetry. We believe that similar results can also be obtained for matrix quasi-exactly solvable models.

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